

Title	Remarks on locally inverse \ast -semigroups (Algorithms in Algebraic Systems and Computation Theory)
Author(s)	Imaoka, Teruo; Fujiwara, Koji
Citation	数理解析研究所講究録 (2002), 1268: 47-49
Issue Date	2002-06
URL	http://hdl.handle.net/2433/42133
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

Remarks on locally inverse \ast -semigroups

島根大学総合理工学部 今岡 輝男 (Teruo Imaoka)
藤原 浩示 (Koji Fujiwara)

Department of Mathematics, Shimane University
Matsue, Shimane 690-8504, Japan

A semigroup S with a unary operation $\ast : S \rightarrow S$ is called a *regular \ast -semigroup* if it satisfies

$$(i) (x^\ast)^\ast = x; \quad (ii) (xy)^\ast = y^\ast x^\ast; \quad (iii) xx^\ast x = x.$$

Let S be a regular \ast -semigroup. An idempotent e in S is called a *projection* if $e^\ast = e$. For a subset A of S , denote the sets of idempotents and projections of A by $E(A)$ and $P(A)$, respectively.

Let S be a regular \ast -semigroup. Define a relation \leq on S as follows:

$$a \leq b \iff a = eb = bf \text{ for some } e, f \in P(S).$$

A regular \ast -semigroup S is called a *locally inverse \ast -semigroup* if eSe is an inverse semigroup for any $e \in E(S)$.

Let G be a non-empty set with a partial product \cdot , a unary operation \ast and a partial order \leq . We simply write ab instead of $a \cdot b$. If ab is defined for $a, b \in G$, we sometimes write $\exists ab$. An element $e \in G$ is called an *idempotent* if $\exists ee$ and $ee = e$. If an idempotent e satisfies $e^\ast = e$, it is called a *projection*. Denote the sets of idempotents and projections of G by $E(G)$ and $P(G)$, respectively.

If G satisfies the following axioms, it is called an *ordered \ast -groupoid*.

(A1) $a(bc)$ exists if and only if $(ab)c$ exists, in which case they are equal.

(A2) $a(bc)$ exists if and only if ab and bc exist.

(A3) $(a^\ast)^\ast = a$.

(A4) If ab exists, then $b^\ast a^\ast$ exists and $(ab)^\ast = b^\ast a^\ast$.

(A5) For any $a \in G$, $a^\ast a$ exists and $a^\ast a$ is the unique projection of G such that $\exists a(a^\ast a)$ and $a(a^\ast a) = a$. We write $a^\ast a = d(a)$ and call it the *domain identity*.

(A6) $a \leq b$ implies $a^\ast \leq b^\ast$.

(A7) For $a, b, c, d \in G$, if $a \leq b$, $c \leq d$, $\exists ac$ and $\exists bd$, then $ac \leq bd$.

(A8) Let $a \in G$ and $e \in P(G)$ such that $e \leq d(a)$. Then there exists a unique element $(a|e)$, called the *restriction* of a to e , such that $(a|e) \leq a$ and $d(a|e) = e$.

(A9) $E(G)$ is an order ideal.

Lemma 1. [3] *Let G be an ordered $*$ -groupoid.*

- (1) *For any $a \in G$, aa^* exists and aa^* is the unique element of $P(G)$ such that $\exists(aa^*)a$ and $(aa^*)a = a$. We write $aa^* = r(a)$ and call it the range identity.*
- (2) *Let $a \in G$ and $e \in P(G)$ such that $e \leq r(a)$. Then there exists a unique element $(e|a)$, called the corestriction of a to e , such that $(e|a) \leq a$ and $r(e|a) = e$.*

An ordered $*$ -groupoid G is called a *locally inductive $*$ -groupoid* if it satisfies

(LG) For any $e, f \in P(G)$, there exists the maximum element in $\langle e, f \rangle = \{(g, h) \in P(G) \times P(G) : g \leq e, h \leq f \text{ and } \exists gh\}$.

Let S be a locally inverse $*$ -semigroup. The representation in [4] raise us a new partial product \cdot on S , which is called a *restricted product*, as follows:

$$a \cdot b = \begin{cases} ab & ab \in R_a \cap L_b \\ \text{undefined} & \text{otherwise} \end{cases}$$

where R_a and L_a denote the \mathcal{R} -class and the \mathcal{L} -class containing a , respectively.

Lemma 2. [3] *$S(\cdot, *, \leq)$ is a locally inductive $*$ -groupoid, which is denoted by $\mathbf{G}(S)$.*

Conversely, let $G(\cdot, *, \leq)$ be a locally inductive $*$ -groupoid. For any $a, b \in G$, there exists the maximum element (e, f) in $\langle d(a), r(b) \rangle = \{(g, h) \in P(S) \times P(S) : g \leq d(a), h \leq r(b), \exists gh\}$. We define a new product \otimes on G as follows:

$$a \otimes b = (a|e)(f|b),$$

and we call it a *pseudoproduct* of a and b .

Lemma 3. [3] *$G(\otimes, *)$ is a locally inverse $*$ -semigroup, which is denoted by $\mathbf{S}(G)$.*

Lemma 4. [3] (1) *For a locally inverse $*$ -semigroup S , we have $\mathbf{S}(\mathbf{G}(S)) = S$.*

(2) *For a locally inductive $*$ -groupoid $G(\cdot, *, \leq)$, we have $\mathbf{G}(\mathbf{S}(G(\cdot, *, \leq))) = G(\cdot, *, \leq)$.*

Let S and T be regular $*$ -semigroups. A mapping $\phi : S \rightarrow T$ is called a *prehomomorphism* if it satisfies

$$(i) \quad (ab)\phi \leq (a\phi)(b\phi),$$

$$(ii) \quad (a\phi)^* = a^*\phi,$$

for all $a, b \in S$.

Lemma 5. [2] *Let S and T be locally inverse $*$ -semigroups and $\phi : S \rightarrow T$ a mapping.*

- (1) *ϕ is a prehomomorphism if and only if it preserves the restricted product and the natural order.*
- (2) *ϕ is a homomorphism if and only if it is a prehomomorphism which satisfies $(ef)\phi = (e\phi)(f\phi)$ for all $e, f \in E(S)$.*
- (3) *The product of prehomomorphisms between locally inverse $*$ -semigroups is also a prehomomorphism.*

A functor between two ordered $*$ -groupoids is said to be *ordered* if it is order-preserving. An ordered functor between two locally inductive $*$ -groupoids is said to be *locally inductive* if it preserves the pseudoproduct.

Now, we have the main result.

Theorem 6. *The category of locally inverse $*$ -semigroups and prehomomorphisms is isomorphic to the category of locally inductive $*$ -groupoids and ordered functors. Moreover, the category of locally inverse $*$ -semigroups and homomorphisms is isomorphic to the category of locally inductive $*$ -groupoids and locally inductive functors.*

References

- [1] Imaoka, T., *Prehomomorphisms on regular $*$ -semigroups*, Mem. Fac. Sci. Shimane Univ. **15** (1981), 23-27.
- [2] Imaoka, T., *Prehomomorphisms on locally inverse $*$ -semigroups*, in: *Words, Semigroups and transductions*, edited by M. Ito, G. Paun and S. Yu, World Scientific, Singapore, 2001, 203-210.
- [3] Imaoka, T. and K. Fujiwara, *Characterization of locally inverse $*$ -semigroups*, Sci. Math. Japon., to appear.
- [4] Imaoka, T. and M. Katsura, *Representations of locally inverse $*$ -semigroups II*, Semigroup Forum **55** (1997), 247-255.
- [5] Lawson, M. V., *Inverse semigroups*, World Scientific, Singapore, 1998.